Tight Bounds on the Relative Performances of Pricing Mechanisms in Storable Good Markets

May 5, 2018

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Abstract

In the storable good monopoly problem, a monopolist sells a storable good by announcing a price in each time period. Each consumer has a unitary demand per time period with an arbitrary valuation. In each period, consumers may buy none, one, or more than one good (in which case the extra goods are stored for future consumption incurring in a linear storage cost). We compare the performance of two important pricing mechanisms on the profitability of the monopolist: pre-announced pricing mechanisms and price contingent mechanisms. In pre-announced pricing the prices in each time period are stated in advance; in a price contingent mechanism each price is stated at the start of the time period, and these prices are dependent upon past purchases. We prove that monopolist can earn at most $O(\log T + \log N)$ times more profit by using a pre-announced pricing mechanism rather than a price contingent mechanism. Here T denotes the number of time periods and N denotes the number of consumers. This bound is tight; examples exist where the monopolist would earn a factor $\Omega(\log T + \log N)$ more by using a pre-announced pricing mechanism.

1 Introduction

The design and analysis of dynamic pricing mechanisms in a monopolistic environment is a fundamental topic in microeconomic theory, and one that has been studied in depth for at least thirty years; see, for example, Roth [1985]. In this setting, the two most studied pricing mechanisms are *pre-announced pricing* (also known as price-commitment) and *contingent*

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pricing (also known as no-commitment pricing, threat pricing, or subgame-perfect pricing). Under pre-announced pricing, the monopolist announces in advance the price at which goods can be purchased at every point in time along the time-horizon. If the monopolist uses a contingent pricing mechanism, the price in a given time period is only announced at the start of that specific period; moreover, the price may depend on the past history. One of the central questions in dynamic pricing is whether pre-announced pricing or contingent pricing provide higher profits for the monopolist. From a mathematical perspective, one difficulty in answering this question is that the tools required to study these mechanisms are quite different. Specifically, the study of pre-announced pricing relies on constrained optimization techniques whereas the study of contingent pricing relies heavily on game theoretic techniques.

In this paper we focus on a model for storable goods, i.e. goods that can be bought and stored for consumption in the future. A confounding feature in pricing storable goods is that a lower price may not only increase the current consumption (*the consumption effect*), but can also induce consumers to store additional goods for future consumption (*the stockpiling effect*).

Dudine et al. [2006] studied a storable good market where a monopolist sells a storable good to consumers with time-dependent demand over an arbitrary number of time periods. A key result they obtained is that consumer surplus and monopolist profits are higher under a pre-announced pricing mechanism than under a price-contingent mechanism. The model of Dudine et al. [2006] assumes that the good is infinitesimally **divisible**, by proposing that there is a continuum of non-atomic buyers or that there is a single consumer in the market who can always obtain some positive additional utility by consuming an additional fraction of the good. Berbeglia et al. [2015] studied the Dudine et al. [2006] model in the setting of **indivisible** goods. Specifically, they analyzed the cases where either there is a finite (possibly very large) number of buyers with a unitary demand per period, or, there is a single buyer with an arbitrary demand per period but who can only obtain value from an integral number of items. Surprisingly, for an indivisible good, in sharp contrast to the case of a divisible good, consumer surplus and monopolist profits may sometimes be lower under a pre-announced pricing mechanism than under a price-contingent mechanism. Indeed, the authors gave a simple two period and two consumer example where profits using a price contingent mechanism were more than 6% higher than could be achieved via a pre-announced pricing mechanism. More generally, Berbeglia et al. [2015] showed that the monopoly profits under a contingent pricing mechanism can be $\Omega(\log T + \log N)$ times more profitable than those obtained under pre-announced pricing mechanism, where T is the number of time periods and N is the number of consumers. Formally, let Π^{CP} and Π^{PA} denote the profits obtained by a contingent pricing mechanism and a pre-announced pricing mechanism, respectively. Then Berbeglia et al. [2015] proved:

Theorem 1.1. *For indivisible storable goods market with N consumers and T time periods, we have* $\Theta(\log T + \log N) \quad \leq_{\exists} \quad \frac{\Pi^{CP}}{\Pi^{PA}} \quad \leq_{\forall} \quad \Theta(\log T + \log N)$

Here $\leqslant_\exists$ means that *there exists* a market instance, with T time periods and N consumers, such that the inequality is satisfied, and $\leqslant_\forall$ means that *for every* instance, with T time periods and N consumers, the inequality is satisfied.

This result suggests the use of a contingent pricing mechanism may be preferable for the monopolist. However, there are several reasons why the monopolist may not wish to use such a mechanism. For example, in practice the use of a threat-based pricing mechanism may not be popular with the firm's customers. In contrast, pre-announced pricing mechanisms are naturally transparent and fair. Furthermore, the optimal strategies in a contingent pricing mechanism are based upon subgame perfect equilibria; whether these actually arise in practice is debatable. Moreover, these equilibria may be very sensitive initial conditions, so they may be less suited for use in incomplete information settings. Again, in contrast, the profitability of pre-announced pricing mechanisms do not change significantly given small changes in the instance, and computing (near)-optimal pre-announced pricing mechanisms can still be straight-forward in many incomplete information settings.

Given this, the aim of this paper if to investigate in more detail the performance of pre-announced pricing mechanisms. As a first step, it turns out that contingent pricing mechanisms do **not** always outperform pre-announced pricing mechanisms. This we can see from the following example which will also serve to illustrate the storable good model (that we define formally in Section 2).

Consumer	**Value at t=1**	**Value at t=2**
I	1	0
II	0	2

Here we have a market with two consumers and two time periods. Consumer I only values the good in period 1 and consumer II only values the good in period 1. There is a storage cost of $\epsilon = 1$, so if the price in period 1 is low enough it may be attractive to the second consumer to purchase the good in period 1 and then store it for consumption in period 2.

In this example the monopolist will benefit from using a pre-announced pricing mechanism instead of a price-contingent mechanism. To see this, let's begin by analyzing the profit obtained under a contingent pricing mechanism. Without loss of generality, throughout this paper, we may assume the monopolist has a cost 0 of producing the item; so the terms revenue and profits are interchangeable. First, suppose the monopolist wants to sell at least one unit in period 1. In that case, the price in period 1 needs to be at most $p_1 = 1$. In this case, consumer I would buy the item and a second item will be also be bought by consumer II. By storing the good for one period, the total cost to consumer II is 2. Consumer II would not be better by waiting until the second period to buy. Because the monopolist does not have to commit to future prices under this mechanism, the optimal price the monopolist would then charge in period 2 is $p_2 = 2$.[1] Thus, the profit of the monopolist is 2. Second, if the monopolist decides not to sell any units in period 1 then consumer I won't buy at all. Further, the period

[1]Ties can be broken in either direction by slightly modifying the price announced.

2 price would be $p_2 = 2$ which would provide the monopolist the same profits. Thus, under a contingent pricing mechanism the profit is $\Pi^{CP} = 2$.

We now show that there exists a pre-announced pricing strategy for the monopolist that can guarantee a profit strictly higher than 2. Suppose the monopolist pre-announces the prices $p_1 = 1$ and $p_2 = 1.99$. Naturally, consumer I would buy at period 1. However, note that consumer II would prefer to buy in period 2 as buying in period 1 and incurring the storage cost of $\epsilon = 1$ is worse than buying at 1.99 in period 2. Thus, the monopolist profit is $2.99 > 2$. So using a pre-announced pricing mechanism provides the monopolist a profit of almost 1.5 times higher than by using a price contingent policy.

This example illustrates that, as well as the practical advantages discussed above, there may be financial incentives for the monopolist to use a pre-announced pricing mechanism. The main contribution of this paper is to quantify exactly how large this financial incentive may be. Specifically, we prove:

Theorem 1.2. *For indivisible storable goods market with N consumers and T time periods, we have*

$$\Theta(\log T + \log N) \quad \leq_{\exists} \quad \frac{\Pi^{PA}}{\Pi^{CP}} \quad \leq_{\forall} \quad \Theta(\log T + \log N)$$

Hence, a pre-announced pricing mechanism can be $\Theta(\log T + \log N)$ times more profitable than a price-contingent mechanism and this bound is tight! Thus, there is a remarkable symmetry with Theorem 1.1. Together, Theorems 1.1 and 1.2 show that, in the storable good monopoly model, neither of the two pricing mechanisms can consistently provide a higher profit for the monopolist. Moreover, whilst the ratios in either direction can be arbitrarily large the performances of the two mechanism are equivalent to within a logarithmic factor.

1.1 Related Literature

The literature on monopoly pricing is extensive. Here, we discuss some of the references most related to our results. One of the most studied monopolistic pricing problems is the durable good monopoly problem.[2] There the monopolist announces a price at each period and unit-demand consumers decide whether to buy at each round or wait. Ronald Coase Coase [1972] conjectured that a durable good monopolist who lacks the power to price commit cannot sell the good above the competitive price. Stokey [1979] showed that the best pre-announced pricing policy consists of setting a fixed price throughout the entire time horizon. The Coase conjecture was proven by Gul et al. [1986] for an infinite time horizon model; they proved the conjecture in a model with a continuum of consumers and where time between periods

[2] A durable can be stored. However, in contrast to the consumable goods studied in this paper, durable goods can be used/consumed repeatedly.

approaches zero. But the Coase conjecture has also been refuted in several models, for example when consumers are atomic (Bagnoli et al. [1989], Berbeglia et al. [2014]) or when there is capacity constraint (McAfee and Wiseman [2008]).

Dasu and Tong [2010] studied pre-announced pricing strategies for a monopoly model in which consumers are atomic and the number of items is limited (i.e. the number of items is less than the number of consumers). They proved that pre-announced pricing is beneficial for the monopolist, but this benefit decreases as the market size increases. Su [2007] considered a monopoly model in a finite time horizon model where buyers arrive continuously and are heterogeneous in patience and valuation. The authors characterized the structure of the optimal contingent pricing policy. Aviv and Pazgal [2008] studied a two-period monopoly model in which consumers arrive under a Poison process. The authors showed that the monopoly profits can increase by switching from a contingent pricing mechanism to a pre-announced pricing policy. Liu and van Ryzin [2008] considered a two-period model of a durable good and analyzed the potential benefit of strategic capacity rationing under pre-announced pricing. Their main result is that rationing can be beneficial for the monopolist but only when consumers are risk-adverse. Correa et al. [2016] studied a new pricing mechanism in which the seller commits to a price menu which states the future price as a function of the available inventory. They proved that there exists a unique equilibrium under this policy when there is a single unit of inventory, but multiple equilibria may exist otherwise. Surasvadi et al. [2017] studied a pricing mechanism in a problem where the monopolist sells a limited number of items over a finite time horizon. The authors found market conditions in which their proposed pricing strategy outperforms a pre-announced pricing. Recently, Ghomi et al. [2018], studied pre-announced pricing mechanisms for a modified version of the storable goods model of Berbeglia et al. [2015] where goods can be stored for a limited time (seasonal goods).

2 The Storable Goods Model for Indivisible Goods

We now describe the model proposed by Berbeglia et al. [2015] for indivisible goods. There is a monopolist facing N consumers with demands for a consumable good over T time periods. The monopolist can produce multiple units of the good at a unitary cost u and wishes to select prices $\{p_1, p_2, \ldots, p_T\}$ to induce sale quantities $\{q_1, q_2, \ldots, q_T\}$ that maximize its profits $\sum_{t=1}^{T} p_t \cdot q_t$. Without loss of generality, we assume that that $u = 0$.

The N consumers seek to maximize their own utilities. Each consumer i has a demand of at most one unit per period and the value she obtains if she consumes the unit at period t is $u_{i,t}$. A consumer i may, however, purchase multiple units $q_{i,t}$ of the indivisible good in any time period and store them for future consumption. There is a storage cost of $\varepsilon \geq 0$ per period and per unit stored. Consequently, the problem for consumer i consists of choosing the consumption quantities $\{x_{i,1}, x_{i,2}, \ldots, x_{i,T}\}$, the number of units purchased per period

$\{q_{i,1}, q_{i,2}, \ldots, q_{i,T}\}$, and the storage levels $\{S_{i,1}, S_{i,2}, \ldots, S_{i,T}\}$. These are chosen to maximize her total utility

$$\sum_{t=1}^{T} (u_{i,t} \cdot x_{i,t} - p_t \cdot q_{i,t} - \varepsilon \cdot S_{i,t})$$

subject to the storage constraints $S_{i,t} = S_{i,t-1} + q_{i,t} - x_{i,t}$, for all $t = 1, \ldots, T$, where $S_{i,0} = 0$. Therefore, the monopolist chooses prices $\{p_1, \ldots, p_T\}$ to maximize $\sum_{t=1}^{T} p_t \cdot q_t$, where $q_t = \sum_i q_{i,t}$ is the total number of items sold in period t.

We distinguish between two fundamental pricing mechanisms: the *pre-announced pricing mechanism* and the *contingent pricing mechanism*. In the pre-announced price mechanism, the monopolist publicly announces in advance (that is, at time $t = 0$) the price schedule $\{p_1, p_2, \cdots, p_T\}$. The consumers can then make their purchasing and storage decisions based upon these prices. In the contingent pricing mechanism the monopolist announces prices sequentially over time. Once the price at period p_t is announced, the consumers must decide how many units to purchase and store without knowledge of future prices. For such a sequential game, the solutions we examine are strong-Markovian[3] pure subgame perfect Nash equilibria (SPNE). We note that the strong-Markovian assumption is standard in literature on sequential pricing problems (e.g. see Ausubel and Deneckere [1989]).

Previous studies have also considered the case where there is a unique buyer (monopsonist) who can consume multiple items of the good in each time period. The results in this paper also extend in a straight forward manner to the single buyer case. We defer discussion of the single buyer case to Appendix A.

We are now ready to quantify exactly how much better pre-announced pricing can do than contingent pricing in this model.

3 Pre-announced Pricing versus Price Contingent Mechanisms

Given a market/game $\mathcal{G}$, let $N(\mathcal{G})$ denote the number of consumers, and let $\Pi^{PA}(\mathcal{G})$ and $\Pi^{CP}(\mathcal{G})$ denote the monopolist's profit under the pre-announced pricing mechanism and the contingent pricing mechanism, respectively, in $\mathcal{G}$. Our goal now is to prove the lower bound in Theorem 1.2. Specifically, we prove:

Theorem 3.1. *There exists an infinite sequence of games $\mathcal{G}_1, \mathcal{G}_2, \ldots$, such that the number of*

[3] That is, consumers decisions are only affected by the current price and the state of the subgame, and the monopolist conditions her pricing strategy only on the payoff-relevant part of the history.

consumers is strictly increasing and that

$$\frac{\Pi^{PA}(\mathcal{G}_i)}{\Pi^{CP}(\mathcal{G}_i)} \;=\; \Omega(\log(N(\mathcal{G}_i)) \;=\; \Omega(\log T) \;=\; \Omega(\log T + \log N)$$

Let's now construct a family of games that imply Theorem 3.1. For every natural number n, we construct an instance $\mathcal{G}(n)$ that has n consumers, each with just a single demand in one period and zero demand in every other period. The consumer unitary demands are all in different periods and the consumption of consumer $i+1$ is earlier than that of consumer i for $i = 1, \ldots, n-1$. Let $\ell_i = \left\lfloor \frac{n^3}{i(i+1)} \right\rfloor$ be the time interval length between the consumption periods of consumer $i+1$ and consumer i. The per unit per period storage cost is defined as $\varepsilon = \frac{1}{n^3}$. Based on the ℓ_i's the number of time periods T is thus

$$T \;=\; \sum_{i=1}^{n-1} \ell_i \;\leqslant\; n^3 \cdot \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) \;=\; n^3 \cdot \frac{n}{n+1} \;\leq\; n^3 \;=\; \frac{1}{\varepsilon}.$$

Let $\tau_i = \sum_{k=i}^{n} \ell_k$, denote the time interval between the consumption periods of consumer i and consumer n. Note that τ_i is also the time period in which agent i has a non-zero demand, τ_i is decreasing in i, and $\tau_1 = T \leq \frac{1}{\varepsilon}$. Notice that we have defined consumer n, (*i.e.* the first consumer in time) to start at time $\tau_n = \Theta(n) > 1$. This is purely for the simplicity of the expressions. The results would be unaffected if we were to shift everything $\Theta(n)$ periods earlier in time.

We now set the consumer valuations: consumer i has a value of $\frac{1}{i}$ at its only consumption time which is τ_i. Observe that higher-valued consumers will consume the good later. Moreover, given the linearity of storage costs, if consumer i purchases the good for a price p at time $t \leqslant \tau_i$, she will receive a utility of $\frac{1}{i} - \varepsilon(\tau_i - t) - p$. We now study how pre-announced pricing and contingent pricing perform in this market.

3.1 Pre-announced Pricing

The following lemma is straightforward and the proof is deferred to Appendix B.

Lemma 3.2. *Suppose the monopolist commits to prices* $(p_1, p_2, \ldots, p_T)$ *where* $p(\tau_i) = \frac{1}{i} - \frac{1}{n^3} \cdot (n - i + 1)$ *for all* $i = 1, \ldots, n$, *and arbitrarily large prices for all other time periods. Then, consumer* i *maximizes its profit by purchasing its unit at time* τ_i *rather than purchasing early and storing.*

Proof. It suffices to prove that consumer $i = 1, \ldots, n-1$ does not benefit from purchasing in period τ_{i+1} and storing it until period τ_i. Since there are ℓ_i time periods between τ_{i+1} and τ_i, we must show that:

$$p_{\tau_i} \leqslant p_{\tau_{i+1}} + \varepsilon \ell_i \tag{1}$$

Now

$$\begin{aligned}
p_{\tau_{i+1}} - p_{\tau_i} + \varepsilon \ell_i &= \frac{1}{i+1} - \frac{1}{n^3}(n-i) - \frac{1}{i} + \frac{1}{n^3}(n-i+1) + \varepsilon \cdot \left\lfloor \frac{1/\varepsilon}{i(i+1)} \right\rfloor \\
&\geqslant \frac{1}{i+1} + \frac{1}{n^3} - \frac{1}{i} + \varepsilon \cdot \left\lfloor \frac{1/\varepsilon}{i(i+1)} \right\rfloor \\
&\geqslant \frac{1}{n^3} - \frac{1}{i(i+1)} + \varepsilon \cdot \left\lfloor \frac{1/\varepsilon}{i(i+1)} \right\rfloor \\
&\geqslant \frac{1}{n^3} \\
&\geqslant 0
\end{aligned}$$

Thus (1) holds. □

We can now analyze the performance of the pre-announced pricing policy.

Theorem 3.3. *There exists a pre-announced pricing strategy which guarantees a profit of at least* $\Pi^{PA} = \Theta(\log T + \log N)$.

Proof. Imagine the monopolist sets the pre-announced prices to be $(p_1, p_2, \ldots, p_T)$ where $p(\tau_i) = \frac{1}{i} - \frac{1}{n^3} \cdot (n-i+1)$ for all $i = 1, \ldots, n$, and has arbitrarily large prices for all other time periods. By Lemma 3.2, consumer i maximizes profit by purchasing its unit at time τ_i, instead of purchasing early and storing.

Then, the monopolist guarantees a profit Π^{PA} of

$$\begin{aligned}
\sum_{i=1}^{n} p_{\tau_i} &= \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{n^3} \cdot (n-i+1) \right) \\
&= \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{n^2} + \frac{i}{n^3} - \frac{1}{n^3} \right) \\
&= \left(\sum_{i=1}^{n} \tfrac{1}{i} \right) - \frac{1}{n} + \frac{n(n+1)}{2n^3} - \frac{1}{n^2} \\
&= \Theta(\log n) + \Theta(\tfrac{1}{n} + \tfrac{1}{n^2}) \\
&= \Theta(\log n)
\end{aligned}$$

Now take $N = n$ and note that the result follows as $\log T = O(\log N)$. □

3.2 Contingent Pricing

Next consider contingent pricing mechanisms for this class of instances. We now show there exists a subgame perfect equilibrium in which the monopolist only makes a profit of $O(1)$. Let $\mathcal{G}_t$ denote a subgame of $\mathcal{G}$ that begins at period time $t \geqslant 0$. In $\mathcal{G}_t$, the number of units of demand at each of the periods $\tau_i \geqslant t$, with $1 \leqslant i \leqslant n$, is either zero (the consumer of that period has already bought) or one. We denote by α_i this zero or one demand at period τ_i. Naturally, the demand for all periods t' with $t' \neq \tau_i$ $(i = 1, \ldots, n)$ is zero. Let $\beta_i = \sum_{j=i}^{T} \alpha_j$ be the total demand remaining after time t'' for all $t'' = t, \ldots, T$.

We now define one strategy for the consumers and another for the monopolist.

The ***restricted-eager*** strategy for the consumers consists of purchasing the unit at the first period t where (i) it would lead to a non-negative profit, and (ii) $t = \tau_j$ for some $j = 1, \ldots, n$.

The ***single-shot*** strategy for the monopolist under a subgame $\mathcal{G}_t$ is defined as follows. The monopolist first computes

$$j^* \quad = \quad \text{argmax}_{j:\tau_j \geqslant t}\ \beta_j \cdot \frac{1}{j}$$

If $t = \tau_{j^*}$ the monopolist then sets the price $\frac{1}{j^*}$. Otherwise, the monopolist charges an arbitrarily high price which we denote as ∞. We note that both the strategies above satisfy the Markovian property, since they depend only on the current time period, the set of remaining buyers, and the currently announced price. We wish to show that this pair of strategies are mutual best-responses. To do this, the following technical lemma will be helpful.

Lemma 3.4. *Suppose the storage cost is $\varepsilon = \frac{1}{n^3}$ and the monopolist sets a price $\frac{1}{i}$ at time τ_i for all (relevant) i. Then if agent i buys her one unit of demand at time τ_j for $j > i$ (i.e. $\tau_j < \tau_i$), her profit is*

$$\pi_i = \sum_{k=i}^{j-1} \sigma_k \quad , \tag{2}$$

where the σ_i's are defined as:

$$\sigma_i \;=\; \frac{1}{i} - \frac{1}{i+1} - \varepsilon \cdot \ell_i \tag{3}$$

Furthermore, $0 \leqslant \pi_i \leqslant \varepsilon \cdot (i - j)$.

Proof. We note that (3) is the difference between the price-savings due to buying early at τ_{i+1} rather than at τ_i, taking storage costs into account. Hence, equation (2) is true by construction. Next note that

$$\sigma_i \;=\; \frac{1}{i} - \frac{1}{i+1} - \varepsilon \cdot \ell_i \;=\; \frac{1}{n^3} \cdot \left(\frac{n^3}{i(i+1)} - \left\lfloor \frac{n^3}{i(i+1)} \right\rfloor \right) \tag{4}$$

Because $0 \leqslant x - \lfloor x \rfloor < 1$ for any $x \in \mathbb{R}$, so (4) implies $0 \leqslant \sigma_i < \frac{1}{n^3} = \varepsilon$. This fact, combined with equation (2) gives us the desired result. □

With this result in hand, we can prove the easier direction of the equilibrium:

Theorem 3.5. ***restricted-eager*** *is a best-response to* ***single-shot****.*

Proof. Let τ_j be the time targeted by the monopolist. The price will be ∞ before τ_j, so the agent cannot afford to buy early. It remains to show that the agent will not benefit from waiting. Suppose that, once τ_j has passed, the monopolist then targets time $\tau_{j'}$. We must have $1 \leqslant j' < j \leqslant n$, and therefore the increase in price is

$$\frac{1}{j'} - \frac{1}{j} \quad \geqslant \quad \frac{1}{j-1} - \frac{1}{j} \quad = \quad \frac{1}{j(j-1)} \quad > \quad \frac{1}{n^2} \quad > \quad \frac{j-j'}{n^3} \quad = \quad \varepsilon \cdot (j - j')$$

However, Lemma 3.4 states that the profit due to buying early is at most $\varepsilon \cdot (j - j')$, which is less than the price increase! Inductively applying this argument to every new target of the monopolist's strategy, we conclude that it is in the agent's (strict) benefit to buy at the original time τ_j. □

It remains to show the converse direction, namely

Theorem 3.6. ***single-shot*** *is a best-response to* ***restricted-eager****.*

We note that the above theorem, and therefore the proof of equilibrium, is a direct corollary of the following lemma since charging a price $\frac{1}{j}$ at time τ_j eliminates all remaining ***restricted-eager*** buyers from the game.

Lemma 3.7. *Suppose that consumers follow* ***restricted-eager****. Then, the optimal strategy for the monopolist at period $t = \tau_j$, for some j, is either to charge a price $\frac{1}{j}$ or to charge a price ∞. For any other t, the monopolist is not worse off by announcing a price of ∞.*

Proof. We will proceed by backwards induction on the starting-time of the subgame. For simplicity of notation, we define some indicators: let $\alpha_i \in \{0, 1\}$ be 1 if and only if the consumer i is present in the current subgame. The base case, *i.e.* the subgame starting (and ending) at τ_1, is straightforward, as either $\alpha_1 \neq 0$, in which case we price $p(\tau_1) = 1$, or there are no remaining agents from whom we can extract profit.

For the induction step, we note that if $t \neq \tau_j$ for any j, the price doesn't affect anything, since agents are playing ***restricted-eager***. Therefore we can set $p(t) = \infty$. Otherwise, if $t = \tau_j$, then, by the induction hypothesis, the optimal strategy in the subgame starting at time $\tau_j + 1$ is `single-shot`. Note that, by Lemma 3.4, consumer i (or consumers of type i) is willing to pay $q_i = \frac{1}{j} + \sum_{k=i}^{j-1} \sigma_k$ at time τ_j, where the σ_i's are defined as in the statement of Lemma 3.4. It should be clear that the optimal price to charge at τ_j will be one of these q_i's. By Lemma 3.4, the q_i's are increasing as i decreases. Hence, setting a price q_i will give a profit $q_i \cdot \sum_{k=1}^{i} \alpha_k$ for the monopolist, and in the subgame starting at $\tau_j + 1$, we will have $\alpha_1 = \cdots = \alpha_{i-1} = \alpha_i = 0$.

Suppose the monopolist decides to set price q_i at time $t = \tau_j$, where $i \leqslant j$ is arbitrary. Let i' be the optimal target of `single-shot` in the subgame starting at time $\tau_j + 1$, if the monopolist charged q_i. We assume $i < j$, otherwise i' is not defined. Note that $\alpha_i = \alpha_{i-1} = \cdots = \alpha_1 = 0$ in the subgame starting at $\tau_j + 1$. The total profit in the subgame starting at time τ_j is therefore

$$q_i \left(\sum_{k=1}^{i} \alpha_k \right) + \frac{1}{i'} \left(\sum_{k=i+1}^{i'} \alpha_k \right) = \left(q_i - \frac{1}{i'} \right) \left(\sum_{k=1}^{i} \alpha_k \right) + \frac{1}{i'} \left(\sum_{k=1}^{i'} \alpha_k \right)$$

Now, since $j > i' \geqslant i$ we have

$$q_i - \frac{1}{i'} \leqslant \frac{1}{j} - \frac{1}{i'} + \varepsilon(i - j + 1) < \frac{i' - j}{n^2} + \frac{n}{n^3} \leqslant 0$$

Hence, the profit due to charging q_i at time τ_j, then playing `single-shot` at the next time, is strictly less than the profit from charging ∞ at τ_j and playing `single-shot`, which is exactly $\frac{1}{i'} \cdot \sum_{k=1}^{i'} \alpha_k$. Therefore, by the induction hypothesis, charging any price other than $q_j = \frac{1}{j}$, or ∞, is suboptimal, and thus the `single-shot` strategy is optimal for the subgame starting at τ_j. □

Theorem 3.8. *There exists a price contingent subgame perfect equilibrium in which the profit of the monopolist is* $\Pi^{CP} = O(1)$.

Proof. By Theorems 3.5 and 3.6, the pair of strategies `restricted-eager` and `single-shot` form a SPNE. It remains then to show that the total profit is at most 1 for any initial subgame. Note that the first nonzero price charged by the monopolist is of the form $\frac{1}{j}$, and at that time, only agents $1, 2, \ldots, j$ remain active. Thus, all the remaining agents have value at least $\frac{1}{j}$. Therefore, all remaining agents will buy, and since there are at most j of them, the total profit is exactly 1 when all players are present. We conclude that for any initial configuration, the profit of the subgame is at most 1. □

4 A Tight Upper Bound

We finish the paper by showing the upper bound in Theorem 1.2.

Theorem 4.1. *For any game $\mathcal{G}$ with N consumers and T periods, we have*

$$\frac{\Pi^{PA}(\mathcal{G})}{\Pi^{CP}(\mathcal{G})} \leq O(\log T + \log N)$$

Proof. Let $\Pi^F(\mathcal{G})$ denote the optimal profit that can be obtained by the monopolist in by setting a fixed price in every period of the game $\mathcal{G}$. Note that $\Pi^F \leq \Pi^{PA}$ as this fixed-price mechanism is a special case of the pre-announced pricing mechanism. We begin by showing that $\Pi^F(\mathcal{G}) \leq \Pi^{CP}(\mathcal{G})$ by contradiction. Take a game $\mathcal{G}$ for which contingent-pricing gives a profit less than $\Pi^F(\mathcal{G})$. Let p^F be the optimal price in the fixed-price mechanism and let $b(i,p) \in \{0,1\}^T$ denote the binary indicator vector such that $(b(i,p))_t = 1$ (i.e. the value at the t^{th} position is one) *if and only if* the value of consumer i in period t satisfies $u_{i,t} \geqslant p$. We claim that under the contingent-pricing mechanism, if the monopolist posts the same price p^F continuously from period 1 until period t, then consumer i will buy at least $\sum_{j=1}^{t}(b(i,p))_j$ units. We prove this by induction. For $t = 1$, take a consumer i such that $u_{i,1} \geq p^F$. Consider the two cases where consumer i does not purchase a unit in period 1 and where consumer i purchases and consumes exactly one unit in period 1. By the strong Markovian assumption, the monopolist will act the same in subgame from period 2 onwards in both of these two cases. Consequently, it is rational for consumer i to buy at least one unit in period 1. Now suppose that the monopolist was announcing the price p^F from period 1 to t. By the induction hypothesis, we know that the number of items ϕ bought by consumer i in periods 1 to $t-1$ satisfies $\phi \geqslant \sum_{j=1}^{t-1}(b(i,p))_j$. If $\phi > \sum_{j=1}^{t-1}(b(i,p))_j$ or if $b(i,p))_t = 0$ the claim holds for period t. So suppose that $\phi = \sum_{j=1}^{t-1}(b(i,p))_j$ and $b(i,p))_t = 1$. This means that consumer i doesn't have any items stored at period t but it has a valuation higher than p^F at period t. Again, as for $t = 1$, by the strong Markovian assumption, the monopolist will act the same in subgame from period $t+1$ onwards regardless on whether consumer i buys and consumes exactly one item at period t or doesn't buy any item at period t at all. It is thus rational for consumer i to buy at least one unit in period t proving the claim. This shows that if the monopolist announces the price p^F in periods 1 through to period t then, regardless of the consumers' off-path beliefs, the number of units sold between period 1 and period t is at least $|\{v_{ij} : v_{ij} \geqslant p^T i = 1, \ldots, N, j = 1, \ldots, t\}|$. Therefore, applying this strategy until the final round T would guarantee the monopolist a profit of at least $\Pi^F(\mathcal{G})$, a contradiction. So $\Pi^F(\mathcal{G}) \leq \Pi^{CP}(\mathcal{G})$. To complete the proof we now show that $\Pi^{PA}(\mathcal{G}) \leq (H_t + H_n) \cdot \Pi^F(\mathcal{G})$, where $H_m = \sum_{i=1}^{m} \frac{1}{i}$ is the m^{th} harmonic number. To do this, let $v_1, v_2, \ldots, v_\ell$ denote, in decreasing order, the set of all consumer valuations over all time periods. The pre-announced pricing mechanism cannot extract more profit than a perfect price discrimination mechanism. Thus $\Pi^{PA}(\mathcal{G}) \leqslant \sum_{k=1}^{\ell} v_k$.

Hence, it suffices to show the following inequality: $\Pi^F(\mathcal{G}) \cdot (H_t + H_n) \geqslant \sum_{k=1}^{\ell} v_k$. Now set $j^* = \arg\max_j\{j \cdot v_j : j \in [\ell]\}$. Thus, if the monopolist announces a fixed price of v_{j^*} then it gets a profit of $j^* \cdot v_{j^*} = \Pi^F(\mathcal{G})$. Without loss of generality, scale the consumer valuations such that $j^* \cdot v_{j^*} = 1$. We then have that, for all $i \in [\ell]$, $i \cdot v_i \leqslant 1$. So $v_i \leqslant \frac{1}{i}$. Given that

$n \cdot T \leqslant \ell$, the inequality follows. □

5 Conclusions

We studied the storable good monopoly model for an indivisible good with respect to two of the most important pricing mechanisms in the dynamic pricing literature: the pre-announced pricing mechanism and the contingent pricing mechanism. Our main result is that there are instances where the monopolist can obtain $O(\log T + \log N)$ times more profit by using a pre-announced pricing mechanism than a contingent pricing mechanism, and this bound is tight. This result suggests that, besides the practical advantages of using a pre-announced pricing mechanism, the financial benefits are also sometimes strong. There are several interesting questions for future research. For example, can one extend our tight bound to the storable good model of Dudine et al. [2006] for a divisible good. Another interesting research direction is to compare both pricing mechanisms for extensions of the storable good model studied here. One such extension is when the goods are storable but are also perishable, i.e. they have an expiration date Ghomi et al. [2018].

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Appendix A: The Single-Buyer Case.

As remarked our results extend to the case where the monopolist faces a unique buyer (monopsonist) for its good. In this setting, N represents the maximum number of items from which the consumer can obtain a positive value on a single period. We formalize this setting as follows. Let the value the buyer receives from consuming x units of the good at period t be given by the function $U(x,t)$. Units are assumed to be indivisible, and therefore the domain of the value function $U(x,t)$ is $\mathbb{N} \times [T]$. Assume the buyer faces a set of prices $\{p_1 \dots, p_T\}$ and that the storage cost is $\varepsilon \geqslant 0$ dollars per unit per time period. The consumer problem then consists of choosing the consumption quantities $\{x_1, x_2, \dots, x_T\}$, the number of units purchased per period $\{q_1, \ldots, q_T\}$, and the storage levels $\{S_1, S_2, \ldots, S_T\}$. These are chosen to maximize her total utility

$$\sum_{t=1}^{T} (U(x_t, t) - p_t \cdot q_t - \varepsilon \cdot S_t)$$

subject to the storage constraints $S_t = S_{t-1} + q_t - x_t$, for all $t = 1, \ldots, T$ where $S_0 = 0$.

The monopolist selects the prices $\{p_1 \ldots, p_T\}$ in order to maximize the resulting profits $\sum_{t=1}^{T} p_t \cdot q_t$. Again, for the pre-announced price mechanism the prices are chosen in advance and for the contingent pricing mechanism the prices are chosen sequentially. In analyzing these two mechanisms we make the following standard assumptions. First, we assume the buyer's value function $U(x, t)$ is non-decreasing in x. Second, we assume the marginal value function is non-increasing in x. Specifically, let $V(x, t) = U(x, t) - U(x - 1, t)$ represent the marginal value of consuming x units of the good at period t. Then we have that $V(x, t)$ is non-increasing in x. We also assume that there exists $H \in \mathbb{N}$ such that $V(H, t) = 0$ for all $t \in T$. No other restrictions are imposed on the consumer valuations. For notational purposes, we set $V(0, t) = L$ for all $t \in [T]$ where L is a huge number.

For motivational purposes the two-consumer example in the introduction can easily be transformed into a single-consumer setting in which the consumer has a value of 1 for period 1 and a value of 2 for period 2.

Furthermore, it is straightforward to extend the proofs in this paper to the single-buyer setting.